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Engineering Notes

Indirect Optimization for Finite-Thrust Time-Optimal Orbital Maneuver

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Introduction

AN IMPORTANT problem in astronautics is to transfer a satellite between elliptic orbits, which has been widely studied by many researchers in both the impulsive case and the continuous low-thrust case [1,2]. Recently, much attention has been focused on the computation of the optimal trajectory for the case of the low-thrust orbital transfer, which can be performed by minimizing the cost of the final time or the fuel-consumption under some additional constraints [2–8].

The dynamics of the satellite are usually described by the position-speed variables or the modified equinoctial elements. Bonnard et al. [3], Caillau and Noailles [4], and Caillau et al. [5] studied the low-thrust time-optimal and minimum fuel-consumption orbital transfer problem using the modified equinoctial elements. The controllability property of the system, the existence of the optimal control and the π-singularity observed in the problem were also proposed from the geometrical analysis viewpoint. In the numerical experiment of the minimum-time transfer problem, some researchers found that the minimum time $t_{f\,\min}$ and the magnitude of the maximal thrust $T_{\max}$ have the relationship that $t_{f\,\min} \times T_{\max} \approx c$, where c is a constant [5,7]. However, the problem of whether there exists a positive constant c such that $t_{f\,\min} \times T_{\max}$ tends to c as $T_{\max}$ tends to zero is still open [8].

The missions of the orbital maneuver include the orbital rendezvous and the orbital intercept, which are different from that of the orbital transfer mainly in the terminal constraint conditions. As early as the 1950s and 1960s, the rendezvous and intercept problems had been widely investigated in the impulsive thrust case [9]. As for the rendezvous problem, the relative dynamics of the satellites (for example, the Hill–Clohessy–Wiltshire equations) have also been concerned [10]. The relative dynamics using the position-speed variables have also been used to deal with the orbital intercept problem [11]. However, the optimization problems for these two maneuver cases are not as widely studied as that for the transfer in the mode of continuous low-thrust using the equations described by the modified equinoctial elements.

In this paper, the optimal-time orbital maneuver problems for the cases of the transfer, intercept, and rendezvous under a unified framework using the modified equinoctial elements are considered. The trajectory optimization problem is reduced to a two-point boundary-value (TPBV) problem by using the Pontryagin Maximum Principle (PMP) [12], and the corresponding terminal conditions for the cases of the orbital transfer, intercept, and rendezvous are studied, respectively.

The main contribution of this paper is twofold. First, the indirect optimization method is applied to the discussion of the finite-thrust orbital intercept and rendezvous problems using the modified equinoctial elements, which is quite different from the previous results on these problems obtained in the impulsive thrust mode and/or by using the relative dynamics of the spacecrafts. To the best knowledge of the authors, this is a novel approach to solve these problems. Second, the numerical simulation results show that the product of the minimum flight time and the maximal thrust is also approximately a constant in the orbital rendezvous and intercept. It is also shown that the orbital transfer and the rendezvous share almost the same optimal trajectory for a fixed maximum magnitude of the thrust.

In this paper, the following notations are used: $\langle , \rangle$ indicates the inner product of two vectors, $|\bullet|$ is the finite-dimensional Euclidean norm, and the superscript T means the transpose of a matrix.

Problem Statement

To give the mathematical formulations of the orbital maneuver, the satellite can be supposed to be modeled as a particle and the high-order terms of the earth's gravitational field and perturbations are neglected, and the propulsion device is a constant specific impulse engine. The 3-D minimum-time orbital maneuver problem of a satellite around the Earth is cast as the following optimal control problem:

$$(TP)_t \begin{cases} \min J = \int_0^{t_f} \mathrm{d}t & \text{performance index} \\ \dot{\boldsymbol{x}} = \boldsymbol{f}_0(\boldsymbol{x}) + \frac{T}{m}\sum_{i=1}^{3} u_i \boldsymbol{f}_i(\boldsymbol{x}) & \text{dymamics of the satellite} \\ \dot{m} = -\beta T & \text{mass flow} \\ \Phi[0, \boldsymbol{x}(0), m(0)] = 0 & \text{initial conditions} \\ \Phi[t_f, \boldsymbol{x}(t_f), m(t_f)] = 0 & \text{terminal conditions} \\ |\boldsymbol{u}| = 1, 0 \le T \le T_{\max} & \text{constraints on control} \end{cases} \tag{1}$$

where β is a constant related to the specific impulse of the engine, $\boldsymbol{u} = [u_1 \quad u_2 \quad u_3]$ is the unit vector in the thrust direction. For a constant specific impulse engine, the controls are defined as the thrust direction and the thrust magnitude as follows:

$$\boldsymbol{u}_c = T\boldsymbol{u}, \qquad 0 \le T \le T_{\max}, \qquad |\boldsymbol{u}| = 1 \tag{2}$$

The modified equinoctial elements $\boldsymbol{x} = [P \quad e_x \quad e_y \quad h_x \quad h_y \quad L]^T$ are used to describe the satellite's motion [2,3]. If the thrust-direction vector is decomposed in the radial-orthoradial frame attached to the satellite, the four vector fields that define the dynamics are [2,5]

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$$f_0 = \sqrt{\frac{\mu}{P}}[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tfrac{W^2}{P}]^T$$

$$f_1 = \sqrt{\frac{P}{\mu}}[0 \quad \sin L \quad -\cos L \quad 0 \quad 0 \quad 0]^T$$

$$f_2 = \sqrt{\frac{P}{\mu}}\frac{1}{W}[2P \quad W\cos L + \eta_x \quad W\sin L + \eta_y \quad 0 \quad 0 \quad 0]^T$$

$$f_3 = \sqrt{\frac{P}{\mu}}\frac{1}{W}[0 \quad -Ze_y \quad Ze_x \quad \tfrac{C}{2}\cos L \quad \tfrac{C}{2}\sin L \quad Z]^T$$

with $W = 1 + e_x \cos L + e_y \sin L$, $\eta_x = e_x + \cos L$, $\eta_y = e_y + \sin L$, $Z = h_x \sin L - h_y \cos L$, and $C = 1 + h_x^2 + h_y^2$, and μ is the gravitation constant.

To show the terminal conditions of the orbital maneuver problem, the Cartesian position (r_1, r_2, r_3) and velocity (v_1, v_2, v_3) of the satellite (in an inertial geocentric reference frame) are given by the modified equinoctial elements as follows [2]:

$$r_1 = \frac{P}{CW}[(1 + h_x^2 - h_y^2)\cos L + 2h_x h_y \sin L] \tag{3a}$$

$$r_2 = \frac{P}{CW}[(1 - h_x^2 + h_y^2)\sin L + 2h_x h_y \cos L] \tag{3b}$$

$$r_3 = \frac{P}{CW}(2Z) \tag{3c}$$

$$v_1 = \frac{1}{C}\sqrt{\frac{\mu}{P}}(2h_x h_y(e_x + \cos L) - (1 + h_x^2 - h_y^2)(e_y + \sin L)) \tag{3d}$$

$$v_2 = \frac{1}{C}\sqrt{\frac{\mu}{P}}((1 - h_x^2 + h_y^2)(e_x + \cos L) - 2h_x h_y(ey + \sin L)) \tag{3e}$$

$$v_3 = \frac{1}{C}\sqrt{\frac{\mu}{P}}(2h_x(e_x + \cos L) + 2h_y(e_y + \sin L)) \tag{3f}$$

Now, for simplicity, the following two assumptions are made:

1) The trajectory is restricted to the elliptic trajectory, and the path constraint $P \geq \Pi^0(\Pi^0 > 0)$ is defined to prevent the satellite from colliding with the earth. Accordingly, to ensure boundedness of the set of admissible trajectory, the trajectory is assumed to stay in a secure zone

$$A = \{x | P \geq \Pi^0, e_x^2 + e_y^2 < 1\}$$

2) In the sequel, the final mass is supposed to be free.

Because the drift f_0 is periodic and the tangent space at any point is spanned by the brackets of $f_0, \ldots, f_3$, no matter how low the thrust might be, the system remains controllable. Thus, the set of admissible trajectories and controls is nonempty and the existence of an optimal control proceeds from the Filippov theorem [13] under the previously stated assumption.

Time-Optimal Control

In this section, the optimal controls for the minimum-time orbital maneuver problem $(TP)_t$ are studied. The PMP is applied and the associated Hamiltonian is

$$H = p_0 + H_0 + \frac{T}{m}\sum_{i=1}^{3} u_i H_i - \beta p_m T + \lambda(|\boldsymbol{u}|^2 - 1) \tag{4}$$

In Eq. (4), p_0 is a nonpositive constant, $H_i(i = 0, 1, 2, 3)$ is the Hamiltonian lift $\langle \boldsymbol{p}, \boldsymbol{f}_i \rangle$, $\boldsymbol{p}$ is the costate vector associated with $\boldsymbol{x}$, p_m is the costate associated with m, and λ is the Lagrange multiplier. In the normal case, p_0 is negative and normalized to -1. Defining $\Psi = [H_1 \quad H_2 \quad H_3]$, the optimal thrust-direction is determined by

$$\begin{cases} \frac{\partial H}{\partial \boldsymbol{u}} = 0 \\ |\boldsymbol{u}| = 1 \\ \frac{\partial^2 H}{\partial \boldsymbol{u}^2} \leq 0 \end{cases} \tag{5}$$

Whenever $\Psi \neq 0$, λ and $\boldsymbol{u}$ can be obtained by

$$\lambda = -\frac{T}{2m}|\Psi| \qquad \boldsymbol{u} = \frac{\Psi}{|\Psi|}$$

Define the thrust switching function as

$$S = \frac{|\Psi|}{m} - \beta p_m \tag{6}$$

Then the Hamiltonian is rewritten to group all of the thrust-dependent terms as follows:

$$H = p_0 + H_0 + S \times T \tag{7}$$

According to the first necessary condition of the PMP, the costate equation on p_m is

$$\dot{p}_m = \frac{T}{m^2}|\Psi|$$

It can be seen that p_m is nonpositive and increasing toward $p_m(t_f)$, which is zero according to the transversality condition and the second assumption listed in Problem Statement. Thus, the switching function S is nonnegative, and the thrust reaches its maximum to maximize the Hamiltonian. One can obtain the following proposition:

Proposition 1: Under the two assumptions listed in Problem Statement, whenever $\Psi = [H_1 \quad H_2 \quad H_3]$ is not equal to zero along an optimal solution, the optimal controls of $(TP)_t$ are given by

$$\boldsymbol{u} = \frac{\Psi}{|\Psi|}, \qquad T = T_{\max} \tag{8}$$

Remark 1: In this paper, the PMP is applied. If the minimum principle is used, p_0 should be normalized to 1, and the optimal controls should be given by $\boldsymbol{u} = -\Psi/|\Psi|$, $T = T_{\max}$. However, the two cases have the same optimal trajectory.

Let $(\boldsymbol{x}, m, \boldsymbol{p}, p_m, \boldsymbol{u})$ be an extremal solution. The classification of regular extremal is based upon the contact order of the trajectory with the switching surface $\{\Psi = 0\}$. The extremal is said to be of order zero if $\boldsymbol{u}$ is smooth and given by Eq. (8) whenever $\Psi \neq 0$, and to be singular if $\Psi \equiv 0$ [3]. It has been proven in [4] that Ψ is continuously differentiable, and the geometric analysis shows that there are only a finite number of switching points in $\{\Psi = 0\}$. Then, one can have the following proposition:

Proposition 2: For the constant specific impulse engine with maximum magnitude of the thrust $T_{\max}$, if there exists an optimal trajectory $(\boldsymbol{x}, m)$ that stays within the interior of the secure zone A, then the corresponding optimal controls of $(TP)_t$ will be such that $|\boldsymbol{u}| = 1$, $T = T_{\max}$ almost everywhere.

According to Proposition 2, $T = T_{\max}$ almost everywhere, the mass can be expressed as a function of time

$$m(t) = m^0 - \beta T_{\max} t \tag{9}$$

and $(TP)_t$ can be given by an equivalent nonautonomous formulation, where m^0 is the initial value of mass of the satellite. Further, for simplicity the problem is recast by scaling the flight time on [0, 1], and treating the final time as an additional constant state variable by letting $\tau = t/t_f$, so $(TP)_t$ can be reduced into the model in the Mayer form as follows:

$$(TP)_\tau \begin{cases} \min J = t_f(1) \\ \dot{\boldsymbol{x}} = t_f(\boldsymbol{f}_0(\boldsymbol{x}) + \frac{T_{\max}}{m(t_f\tau)} \sum_{i=1}^{3} u_i \boldsymbol{f}_i(\boldsymbol{x})), \tau[0,1] \\ \dot{t}_f = 0 \\ \Phi(0, \boldsymbol{x}(0), t_f(0)) = 0, \Phi(1, \boldsymbol{x}(1), t_f(1)) = 0 \\ |\boldsymbol{u}| = 1 \end{cases} \tag{10}$$

If $(\boldsymbol{x}, t_f, u)$ is the solution of $(TP)_\tau$, there will be absolutely continuous costates $\boldsymbol{p} = [\, p_p \quad p_{e_x} \quad p_{e_y} \quad p_{h_x} \quad p_{h_y} \quad p_L \,]^T$ and p_{t_f} associated to $\boldsymbol{x}$ and t_f, respectively, such that $(\boldsymbol{x}, t_f, \boldsymbol{p}, p_{t_f})$ is a solution of the following two-point boundary-value problem obtained from the first-order necessary condition of the PMP:

$$\dot{\boldsymbol{x}} = \frac{\partial H^*}{\partial \boldsymbol{p}} \tag{11a}$$

$$\dot{t}_f = 0 \tag{11b}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial H^*}{\partial \boldsymbol{x}} \tag{11c}$$

$$\dot{p}_{t_f} = -\frac{\partial H^*}{\partial t_f} \tag{11d}$$

with initial boundary conditions

$$\Phi(0) = (P(0) - P^0, e_x(0) - e_x^0, e_y(0) - e_y^0, h_x(0) - h_x^0, h_y(0) - h_y^0, L(0) - L^0, p_{t_f}(0) - p_{t_f}^0) = 0 \tag{12}$$

and terminal boundary conditions

$$\Phi(1) = \Phi[1, \boldsymbol{x}(1), t_f(1), \boldsymbol{p}(1), p_{t_f}(1)] = 0 \tag{13}$$

which for different cases are determined by different missions of the orbital maneuver. Also, in Eq. (11)

$$H^* = \left\langle \boldsymbol{p}, t_f \left(\boldsymbol{f}_0(\boldsymbol{x}) + \frac{T_{\max}}{m(t_f\tau)} \sum_{i=1}^{3} u_i^* \boldsymbol{f}_i(\boldsymbol{x}) \right) \right\rangle \tag{14}$$

is the Hamiltonian of the $(TP)_\tau$, where the optimal control $\boldsymbol{u}^* = [\, u_1^* \quad u_2^* \quad u_3^* \,]$ is still defined as a smooth function in Eq. (8).

Now, the continuous-thrust minimum-time orbital maneuver is reduced into a two-point boundary-value problem. In the next section, the terminal boundary conditions for the various orbital maneuver missions will be discussed.

Boundary-Value and Transversality Conditions

The following definitions are made to describe the different types of trajectory that can be specified in a particular orbital maneuver problem [9]:

1) Transfer: starts from a prescribed initial motion and ends at a prescribed final motion, for example, orbit-to-orbit.

2) Intercept: starts from a prescribed or a partially prescribed initial condition and ends at a partially prescribed final condition, for example, from a circular orbit to a specified radius and path angle as in some disorbit problems.

3) Rendezvous: starts from a prescribed initial motion and ends at a time-related prescribed final motion, for example, satellite-to-satellite.

In this paper, the various missions of orbital maneuver such as transfer, intercept, and rendezvous are supposed to have common initial boundary conditions Eq. (12), where $p_{t_f}^0 = 0$ by transversality, and the main difference in these three orbital maneuver missions is the terminal constraint conditions. In the sequel, $(P^f, e_x^f, e_y^f, h_x^f, h_y^f, L^f)$ are used to denote the final orbit of the satellite or the virtual object, and the terminal constraints of transfer, intercept, and rendezvous maneuver missions are discussed in detail, respectively.

Orbital Transfer Problem

The orbital transfer problem has been widely studied, and the terminal boundary conditions meet [3,4]

$$(P(1) - P^f, e_x(1) - e_x^f, e_y(1) - e_y^f, h_x(1) - h_x^f, h_y(1) - h_y^f) = 0$$

and $[p_L(1) - p_L^f, p_{t_f}(1) - p_{t_f}^f] = 0$. Here $p_L^f = 0$, $p_{t_f}^f = -1$ due to the transversality condition of the PMP with free final longitude and Mayer form performance index t_f. So for the orbital transfer problem, the terminal boundary constraint is defined by

$$\Phi(1) = (P(1) - P^f, e_x(1) - e_x^f, e_y(1) - e_y^f, h_x(1) - h_x^f, h_y(1) - h_y^f, p_L(1), p_{t_f}(1) + 1) = 0 \tag{15}$$

Remark 2: In fact, the fixed final longitude means that both the position of the satellite on the final orbit and the number of revolutions are fixed. This case can be considered as the orbital rendezvous provided that the flight time t_f is also fixed [14]. However, in the minimum-time orbital maneuver problem, the longitude L is a fast variable with time. So the final longitude that is related to the maximum thrust $T_{\max}$ cannot be chosen freely. The product of the minimum longitude and the maximal thrust is nearly a constant [2,6]. Therefore, the problem of the minimum transfer time with a fixed final longitude cannot be simply considered as a rendezvous.

Orbital Intercept Problem

The orbital intercept problem should be considered under the conditions that the relative position of two satellites is zero while the relative velocity is free at the terminal time. The two satellites are called the interceptor and the target, respectively, and the target is supposed to move on a fixed orbit without thrust. For a given targeted satellite, the terminal condition of the interceptor is not a fixed orbit but a manifold satisfying the following constraints:

$$\phi_1 \doteq r_1(1) - r_1^f = 0 \tag{16a}$$

$$\phi_2 \doteq r_2(1) - r_2^f = 0 \tag{16b}$$

$$\phi_3 \doteq r_3(1) - r_3^f = 0 \tag{16c}$$

where (r_1^f, r_2^f, r_3^f) is the position vector of the target which can be obtained by substituting $(P^f, e_x^f, e_y^f, h_x^f, h_y^f, L^f)$ into Eqs. (3a–3c). However, for a given initial position, the final position on the orbit of the target is not fixed if the flight time is not known in advance. The longitude L^f varies with time without thrust as

$$\frac{dL^f}{d\tau} = t_f \sqrt{\frac{\mu}{P^f}} \frac{(1 + e_x^f \cos L^f + e_y^f \sin L^f)^2}{P^f} \tag{17}$$

For the convenience of the discussion, a new variable L^f described by Eq. (17) is introduced into $(TP)_\tau$. So the state vector is $(\boldsymbol{x}, t_f, L^f) \in \mathbb{R}^6 \times \mathbb{R} \times \mathbb{R}$, and the equation $L^f(0) - L^{f^0} = 0$ is added into Eq. (12) as another constraint of the initial conditions. By virtue of the PMP, the conditions of transversality are determined from the terminal constraints Eq. (16) as follows:

$$p_P(1) = -\left(\lambda_1 \frac{\partial \phi_1}{\partial P(1)} + \lambda_2 \frac{\partial \phi_2}{\partial P(1)} + \lambda_3 \frac{\partial \phi_3}{\partial P(1)} \right) \tag{18a}$$

$$p_{e_x}(1) = -\left(\lambda_1 \frac{\partial \phi_1}{\partial e_x(1)} + \lambda_2 \frac{\partial \phi_2}{\partial e_x(1)} + \lambda_3 \frac{\partial \phi_3}{\partial e_x(1)} \right) \tag{18b}$$

$$p_{e_y}(1) = -\left(\lambda_1 \frac{\partial \phi_1}{\partial e_y(1)} + \lambda_2 \frac{\partial \phi_2}{\partial e_y(1)} + \lambda_3 \frac{\partial \phi_3}{\partial e_y(1)} \right) \tag{18c}$$

$$p_{h_x}(1) = -\left(\lambda_1 \frac{\partial \phi_1}{\partial h_x(1)} + \lambda_2 \frac{\partial \phi_2}{\partial h_x(1)} + \lambda_3 \frac{\partial \phi_3}{\partial h_x(1)} \right) \tag{18d}$$

Table 1 Boundary conditions

Variables	Initial condition	Terminal condition
P	15.6 Mm	32.1 Mm
e_x	0.75	0.16
e_y	0.0	0.30
h_x	0.612	0.0
h_y	0.0	0.1
L	π	free
L^f	1.507	free

$$p_{h_y}(1) = -\left(\lambda_1 \frac{\partial \phi_1}{\partial h_y(1)} + \lambda_2 \frac{\partial \phi_2}{\partial h_y(1)} + \lambda_3 \frac{\partial \phi_3}{\partial h_y(1)}\right) \tag{18e}$$

$$p_L(1) = -\left(\lambda_1 \frac{\partial \phi_1}{\partial L(1)} + \lambda_2 \frac{\partial \phi_2}{\partial L(1)} + \lambda_3 \frac{\partial \phi_3}{\partial L(1)}\right) \tag{18f}$$

$$p_{L_f}(1) = \lambda_1 \frac{\partial r_1^f}{\partial L^f(1)} + \lambda_2 \frac{\partial r_2^f}{\partial L^f(1)} + \lambda_3 \frac{\partial r_3^f}{\partial L^f(1)} \tag{18g}$$

where $\lambda_i (i = 1, 2, 3)$ is the Lagrange multiplier.

The multiplier λ_i can be determined by solving any three equations of Eq. (18), then four terminal constraints are obtained by substituting λ_i into the remaining four equations of Eq. (18). So the four terminal constraints, Eq. (16) and $p_{t_f}(1) = -1$ form eight terminal conditions of the orbital intercept problem.

Orbital Rendezvous Problem

Different from the orbital intercept problem, for the orbital rendezvous problem both the relative position and the relative velocity are required to be zero at the terminal time, that is

$$\phi_1 \doteq r_1(1) - r_1^f = 0 \tag{19a}$$

$$\phi_2 \doteq r_2(1) - r_2^f = 0 \tag{19b}$$

$$\phi_3 \doteq r_3(1) - r_3^f = 0 \tag{19c}$$

$$\phi_4 \doteq v_1(1) - v_1^f = 0 \tag{19d}$$

$$\phi_5 \doteq v_2(1) - v_2^f = 0 \tag{19e}$$

$$\phi_6 \doteq v_3(1) - v_3^f = 0 \tag{19f}$$

Similarly, where (r_1^f, r_2^f, r_3^f) and (v_1^f, v_2^f, v_3^f) are the position and velocity vectors of the target which can be obtained by substituting $(P^f, e_x^f, e_y^f, h_x^f, h_y^f, L^f)$ into Eqs. (3a–3f) Furthermore, L^f described by Eq. (17) is also introduced into $(TP)_\tau$ as a new state variable, and by transversality, the costate $\boldsymbol{p} = [\,p_p \quad p_{e_x} \quad p_{e_y} \quad p_{h_x} \quad p_{h_y} \quad p_L]^T$ meets the terminal constraint as follows

$$\boldsymbol{p}\,(1) = -\sum_{i=1}^{6} \lambda_i \frac{\partial \phi_i}{\partial \boldsymbol{x}(1)} \tag{20}$$

By solving the algebraic Eq. (20), the Lagrange multipliers λ_i, $i = 1, \cdots, 6$ can be attained. Then substituting them into the following terminal constraint equation on p_{L^f}

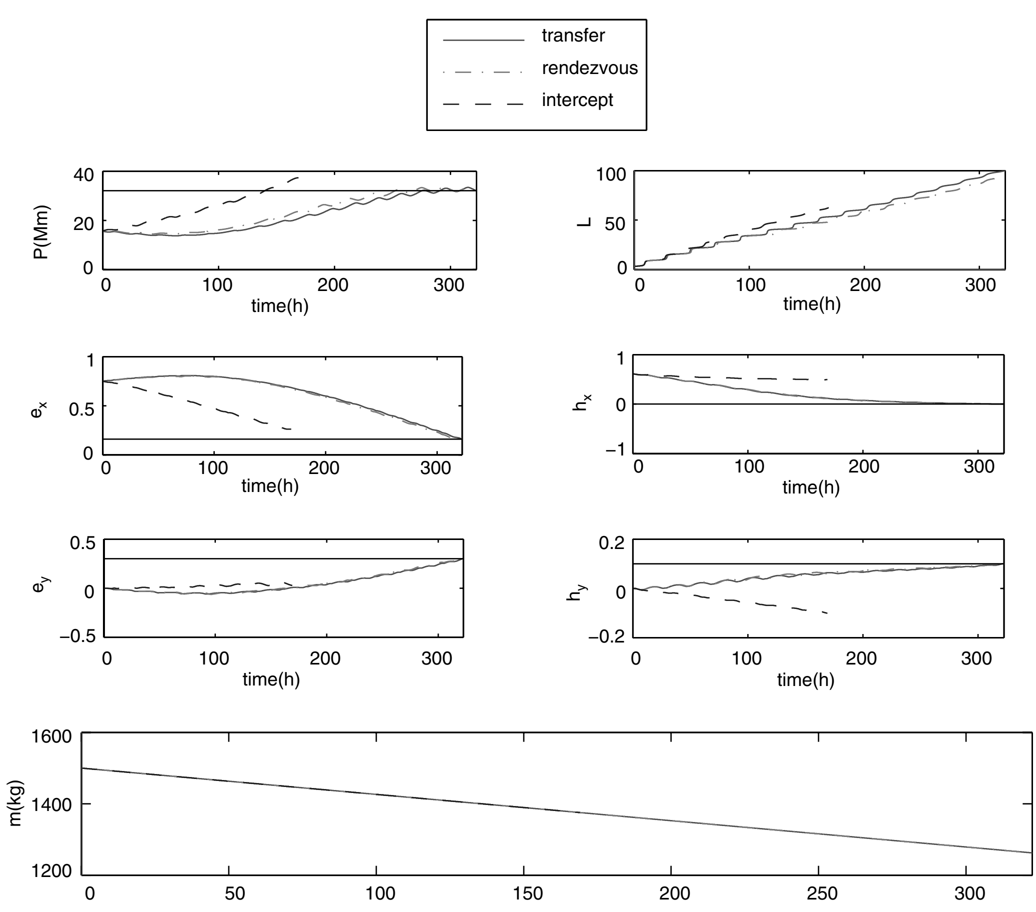

Fig. 1 Optimal solution of the states, thrust of 4 N.

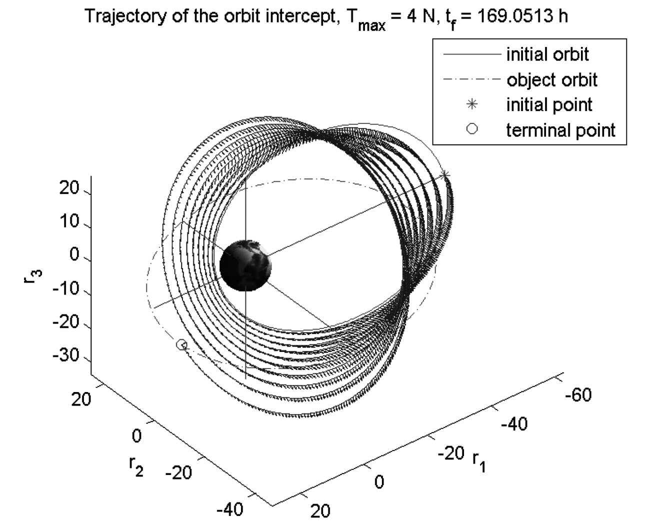

Fig. 2 Optimal 3-D trajectory of the orbital intercept, thrust of 4 N.

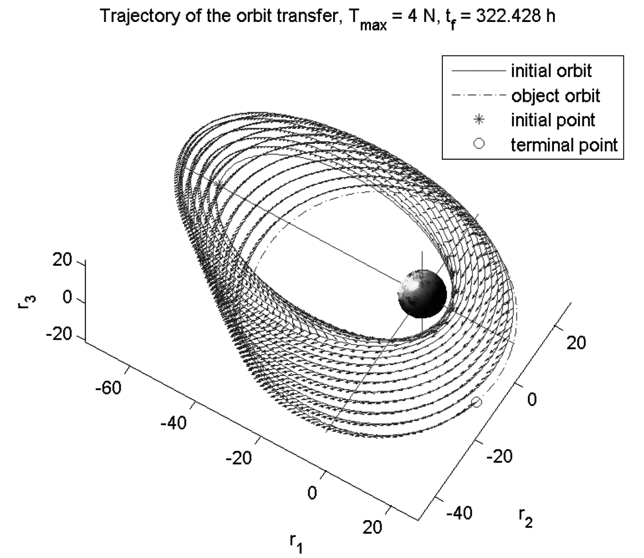

Fig. 3 Optimal 3-D trajectory of the orbital transfer, thrust of 4 N.

$$p_{L^f}(1) = \lambda_1 \frac{\partial r_1^f}{\partial L^f(1)} + \lambda_2 \frac{\partial r_2^f}{\partial L^f(1)} + \lambda_3 \frac{\partial r_3^f}{\partial L^f(1)} + \lambda_4 \frac{\partial v_1^f}{\partial L^f(1)} + \lambda_5 \frac{\partial v_2^f}{\partial L^f(1)} + \lambda_6 \frac{\partial v_3^f}{\partial L^f(1)} \tag{21}$$

One can get another terminal condition. So, one can obtain eight terminal conditions of the orbital rendezvous problem with Eqs. (19) and (21) and $p_{t_f}(1) = -1$.

Numerical Simulation

In this section, the numerical simulations of the orbital maneuver problems for the cases of the transfer, intercept, and rendezvous are presented. In the previous section, the continuous-thrust time-optimal orbital maneuver problem is reduced into the TPBV problem by applying the PMP. In this section, the single shooting method is used to solve the corresponding TPBV problem. Taking the orbital transfer problem as an example, the boundary-value problem is equivalent to finding zeros of the so-called shooting function, that is, finding $(\boldsymbol{p}^0, t_f^0) \in \mathbb{R}^6 \times \mathbb{R}$ such that

$$b(\Phi^0(\boldsymbol{x}^0, t_f^0, \boldsymbol{p}^0, p_{t_f}^0)) = 0 \tag{22}$$

where the boundary function b is defined by Eq. (15).

In a numerical example, the physical constants in the system $(TP)_\tau$ are [3]

$$\mu = 5165.8620912\ \mathrm{Mm^3/h^2} \qquad \beta = 1.42 \times 10^{-2}\ \mathrm{h/Mm}$$
$$m^0 = 1500\ \mathrm{kg}$$

and the boundary conditions are summarized in Table 1, where $1\ \mathrm{Mm} = 10^6\ \mathrm{m}$.

First, the solutions of the states for the orbital transfer, intercept, and rendezvous problems with the maximum thrust of 4 N are shown in Fig. 1. The graphs show the six orbital elements $(P, e_x, e_y, h_x, h_y, L)$ and the mass m of the satellite with time as the abscissa, and the solid line, dashed-dot line and dashed line represent the orbital transfer, rendezvous, and intercept, respectively. The values of the targeted orbital elements are also shown by the horizontal solid line in the corresponding subfigures. For the maximum thrust of 4 N, the minimum flight time is 322.428 h for the transfer problem, 318.986 h for the rendezvous problem, and 169.0513 h for the intercept problem, respectively.

It can be seen from Fig. 1 that the evolution of the state variables is quite smooth due to the use of the modified equinoctial orbital elements, and also to the continuous low-thrust of the propulsion. It can also be seen that the states of the transfer problem and the rendezvous problem share almost the same trajectory, and the final values of the elements in the orbital rendezvous also reach the desired orbital elements as those in the orbital transfer. However, the final values of the elements in the orbital intercept problem are much different from those of the elements of the object orbit, which also can be seen from Fig. 2, in that the final orbit of the interceptor is nearly perpendicular to the orbit of the target.

Figure 2 shows the 3-D optimal trajectory in (r_1, r_2, r_3) of the orbital intercept problem, and the arrows picture the action of the control. The solid line denotes the orbit of the interceptor, and the dashed-dot line denotes the orbit of the target. The intercept point is labeled as a circle. From Fig. 2 one can see that the change in the inclination is less than that in the orbital transfer problem shown in Fig. 3. Because the orbital rendezvous problem shares almost the same trajectory as that of the orbital transfer problem, the trajectory of the rendezvous problem is not shown here.

Table 2 Minimum flight time

$T_{\max}$, N	Transfer case		Rendezvous case		Intercept case	
	$t_{f\min}$, h	$\lvert b\rvert$	$t_{f\min}$, h	$\lvert b\rvert$	$t_{f\min}$, h	$\lvert b\rvert$
0.5	2556.1677	1.1E − 07	2554.3085	4.6E − 09	1354.1545	2.1E − 09
1.0	1279.796	2.7E − 11	1260.9954	3.4E − 09	685.094	4.7E − 11
1.5	–	–	842.5645	2.9E − 09	436.7368	5.5E − 11
2.0	635.9953	6.1E − 12	632.504	5.9E − 09	–	–
4.0	323.9285	1.7E − 12	318.9862	8.3E − 10	169.0513	3.1E − 12
6.0	216.1717	2.3E − 12	213.0968	6.3E − 10	111.084	4.6E − 12
10	128.3579	2.4E − 12	127.6397	5.4E − 11	73.542	4.0E − 12
20	63.8795	5.2E − 13	65.8506	5.1E − 12	34.6528	5.1E − 13
40	34.9649	1.3E − 12	33.217	7.7E − 12	21.469	7.2E − 13
60	25.0451	1.1E − 13	28.2547	1.8E − 12	20.4102	4.7E − 13

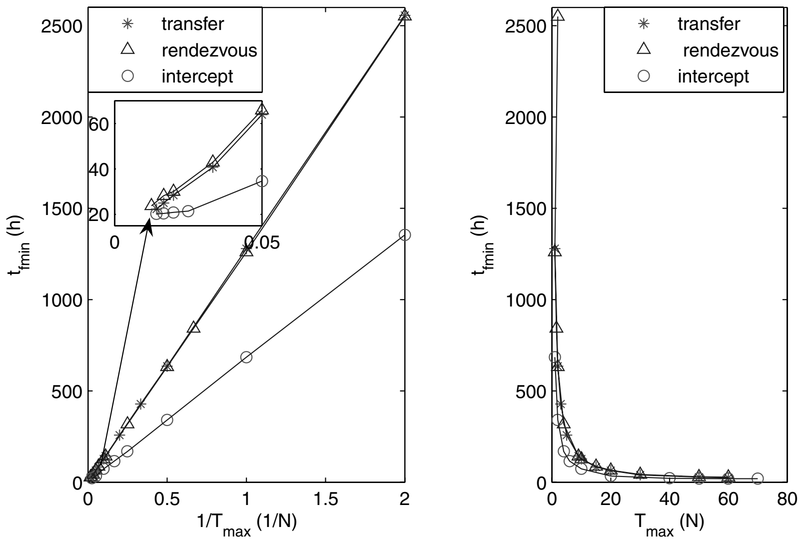

Fig. 4 Near constancy of the product $t_{f\,\min} \times T_{\max}$.

Adjusting the magnitude of the maximal thrust and repeating the numerical experiment, the results of the minimum flight time in the three maneuver cases are summarized in Table 2. The relationships between the minimum flight time and the maximal thrust are pictured in Fig. 4. From the left graph of this figure, one can see that the minimum flight time of the orbital transfer and that of the rendezvous problem are nearly equal with the same fixed maximum thrust $T_{\max}$, and the minimum time of the orbital intercept is less than that of the orbital transfer or rendezvous. Though some researchers found that the minimum time and the modulus of the maximal thrust have approximatively the relationship $t_{f\,\min} \times T_{\max} \approx c$ [5,7], the conjecture seems to be also suitable for the orbital intercept and rendezvous problems. However, one can also see from the magnified local image of the left graph and the right graph that the relationship does not hold for the case with high thrust. The minimum flight time does not decrease significantly with the increasing of the magnitude of the maximal thrust when the thrust is greater than a certain magnitude (for example 60 N). Then it becomes natural to suppose the impulsive thrust as the limit of the continuous finite thrust when the magnitude of the maximum thrust grows, but this has not been theoretically proven. Moreover, the maximum thrust of 60 N is not realistic for a 1500 kg satellite, and it is only used for drawing the previously mentioned conclusion in the numerical simulation.

Conclusions

The continuous finite-thrust minimum-time orbital maneuver missions have been considered in this paper, which include the transfer, intercept, and rendezvous. The modified equinoctial elements are used to describe the dynamics of the satellite, and the terminal constraints of the three maneuver missions are studied, respectively, by the conditions of transversality. Under a unified theoretical framework, the time-optimal maneuver is reduced into the corresponding two-point boundary-value problem by the Pontryagin Maximum Principle.

Numerically the single shooting method is applied, which has been proven to be efficient for numerical experiments. The simulation results demonstrate: 1) for the same fixed maximum thrust, the orbital transfer and rendezvous problems share almost the same optimal trajectory, 2) for the same maximum thrust, the minimum flight time of the intercept mission is less than that of the transfer or rendezvous problem, and 3) the conjecture that the product of the minimum time and the magnitude of the maximal thrust is nearly a constant also fits for the intercept and rendezvous problems, but one also sees that the relationship is not tenable in the range of high thrust.

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